Lecture : Simplifying the BP decoder (contd.) and a discussion on density evolution

We shall first briefly discuss how the simplification for the check $\to$ variable messages came about.

Recall that each factor node computes:

$$\nu_{a\to i}^{(t+1)}(x_i) = \sum_{x_{N(a)\setminus\{i\}}} \mathbb{1}\{\bigoplus_{j\in N(a)} x_j = 0\} \cdot \prod_{j\in N(a)\setminus\{i\}} \mu_{j\to a}^{(t)}(x_j) \quad \text{—— (A)}$$

Let $|N(a)\setminus\{i\}| = d$ and consider any enumeration $1,\ldots,d$ of the elements in $N(a)\setminus\{i\}$. Then, let

$$\phi(z) = \prod_{j=1}^{d}\left(\mu_{j\to a}^{(t)}(0) + z\,\mu_{j\to a}^{(t)}(1)\right)$$

$$= \sum_{x_1^d\in\{0,1\}^d} \prod_{j=1}^{d} \mu_{j\to a}^{(t)}(x_j)\, z^{w(x_1^d)},$$

for $z \in \mathbb{R}$.

We then have that

$$\frac{1}{2}\left[\phi(1)+\phi(-1)\right] = \sum_{x_1^d\in\{0,1\}^d}\left[\prod_{j=1}^{d}\mu_{j\to a}^{(t)}(x_j)\right]\left(\frac{1}{2}\left(1^{w(x_1^d)} + (-1)^{w(x_1^d)}\right)\right)$$

$$= \sum_{x_1^d\in\{0,1\}^d} \prod_{j=1}^{d} \mu_{j\to a}^{(t)}(x_j)\cdot \mathbb{1}\{\bigoplus_{j=1}^{d} x_j = 0\},$$

which is precisely the update in ⊛.

Quick HW: Can you derive a similar expression for

$$\sum_{x^d \in \{0,1\}^d} \prod_{j=1}^{d} \mu_{j\to a}^{(t)}(x_j) \cdot \mathbb{1}\left\{\bigoplus_{j=1}^{d} x_j = 1\right\} ?$$

Hence, we obtain that

$$L_{a\to i}^{(t+1)} = \ln \frac{\prod_{j=1}^{d}\left(\mu_{j\to a}^{(t)}(0)+\mu_{j\to a}^{(t)}(1)\right)+\prod_{j=1}^{d}\left(\mu_{j\to a}^{(t)}(0)-\mu_{j\to a}^{(t)}(1)\right)}{\prod_{j=1}^{d}\left(\mu_{j\to a}^{(t)}(0)+\mu_{j\to a}^{(t)}(1)\right)-\prod_{j=1}^{d}\left(\mu_{j\to a}^{(t)}(0)-\mu_{j\to a}^{(t)}(1)\right)}$$

$$= \ln \frac{1+\prod_{j=1}^{d}\tanh\left(\frac{1}{2}L_{j\to a}^{(t)}\right)}{1-\prod_{j=1}^{d}\tanh\left(\frac{1}{2}L_{j\to a}^{(t)}\right)}$$

[since $\tanh\left(\frac{1}{2}\ln\frac{a}{b}\right) = \frac{a-b}{a+b}$]

$$= 2\tanh^{-1}\left(\prod_{j=1}^{d}\tanh\left(\frac{1}{2}L_{j\to a}^{(t)}\right)\right),$$

[since $2\tanh^{-1}(z) = \ln\frac{1+z}{1-z}$]

thereby verifying the simplification of the computation.

A discussion on density evolution

We end our discussion on LDPC codes with a brief discussion on analyzing the performance of the BP decoder for a certain class of LDPC codes, over the BEC(ϵ).

It can be verified that the message-passing routine over the BEC(ϵ) simply reduces to the peeling decoder we had discussed earlier.

In practice, irregular ensembles of LDPC codes are used instead of the regular ensemble we had seen earlier.

<u>Def 1</u>: The standard irregular ensemble LDPC(Λ, P) of LDPC codes is obtained by first picking Λ_i variable nodes (resp. P_i check nodes) of degree i, $1 \le i \le l_{max}$ (resp. $1 \le i \le r_{max}$), giving rise to the generating functions $\Lambda(x) = \sum_{i=1}^{l_{max}} \Lambda_i x^i$ and $P(x) = \sum_{i=1}^{r_{max}} P_i x^i$.
Let the no. of variable nodes (resp. check nodes) be n (resp. m).
The ensemble is then generated by picking a random permutation to connect the half-edges at the variable and check nodes, as earlier.

<u>HW</u>: ① Verify that $\Lambda(1) = n$ and $P(1) = m$.
② Define $L(x) = \dfrac{\Lambda(x)}{\Lambda(1)}$ and $R(x) = \dfrac{P(x)}{P(1)}$.

Further, define $\lambda(x) = \dfrac{L'(x)}{L'(1)}$ and $\rho(x) = \dfrac{R'(x)}{R'(1)}$.

(i) Show that $L(x) = \dfrac{\int_0^x \lambda(s)\,ds}{\int_0^1 \lambda(s)\,ds}$ and $R(x) = \dfrac{\int_0^x \rho(s)\,ds}{\int_0^1 \rho(s)\,ds}$.

(ii) What is $\Lambda'(1)$ and $P'(1)$?

③ Let the designed rate be $r = 1 - \frac{m}{n} = 1 - \frac{P(1)}{\Lambda(1)}$ (from ①).

Prove that $r = 1 - \dfrac{L'(1)}{R'(1)} = 1 - \dfrac{\int_0^1 \rho(s)\,ds}{\int_0^1 \lambda(s)\,ds}$.

Now, for fixed degree distributions from the edge perspective λ and ρ, consider the ensemble $LDPC(n,\lambda,\rho)$ defined as follows: from λ, ρ, obtain Λ, R, r (see HW above) and pick a suitable n such that nL_i and $nr(1-R_j)$ are integers, for all admissible i,j. For such n, the ensemble $LDPC(n,\lambda,\rho)$ equals the ensemble $LDPC(n\Lambda, nR)$.

FACT: For an LDPC code chosen uniformly at random from the ensemble $LDPC(n,\lambda,\rho)$, we have that the "computation graphs" of fixed height, say $2l$, obtained by unrolling the edges "revealed" during message-passing iterations, is a tree w.p. 1 as $n \to \infty$.

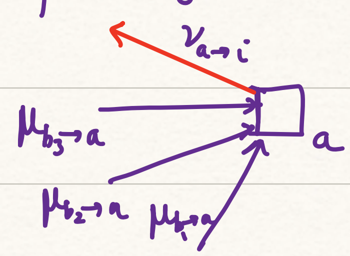
[Recall that this is good, since as argued in the last lecture, message-passing is guaranteed to converge to the correct marginals, if the factor graph is a tree.]

The above fact allows us to focus on cycle-free LDPC codes in $LDPC(n,\lambda,\rho)$.
We now provide a short sketch of the density evolution analysis over the $BEC(\epsilon)$, for such a code.

- Let $p_0 \triangleq \epsilon$, the channel erasure probability
- We aim to find a recursive formula for p_ℓ, $\ell \geq 1$, which is the probability that a random edge (drawn $\sim \lambda$ or ρ) carries a variable $\to$ check erasure in round ℓ of MP decoding ($\equiv$ peeling decoder operation).

- We then aim to find the largest ϵ, called ϵ^{BP} (or BP threshold) such that $p_\ell \xrightarrow[\ell\to\infty]{} 0$.

① Note that for a fixed edge e:

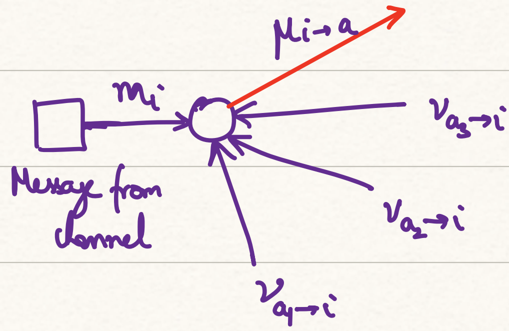


$$\Pr[\nu_{a\to i} = ? \mid e = (i,a)]$$
$$= \Pr[\mu_{b_j\to a} = ?\ ,\text{ for some } j = 1,\dots,d-1]$$
$$= 1-(1-p_{\ell-1})^{d-1}$$

Now, since the edge is randomly chosen from the distribution ρ, we have

$$\Pr[\nu_{a\to i} = ?] = \sum_{d=1}^{r_{max}} \rho_d \cdot \left(1-(1-p_{\ell-1})^{d-1}\right) = 1-\underbrace{\rho(1-p_{\ell-1})}_{\rho(x)\text{ at }1-p_{\ell-1}}.$$

② Next, fix an edge emanating from a variable node:



$$\Pr[\mu_{i\to a}=?\mid e=(i,a)]$$
$$=\epsilon\left(1-\rho(1-p_{\ell-1})\right)^{d-1}$$
$$=p_0\left(1-\rho(1-p_{\ell-1})\right)^{d-1}$$

[all incoming messages are erasures]

Again, since the edge is drawn from λ, we obtain that

$$p_\ell = \Pr[\mu_{i\to a}=?] = p_0\,\lambda\left(1-\rho(1-p_{\ell-1})\right)$$

Density Evolution (DE) recursion

<u>FACT</u>: There exists a threshold $\epsilon^{BP}=\epsilon^{BP}(\lambda,\rho)$ such that for $\epsilon<\epsilon^{BP}$, we have $\lim_{\ell\to\infty} p_\ell = 0$. Further, let $\epsilon(x)=\dfrac{x}{\lambda(1-\rho(1-x))}$ be the DE fixed point.

We have that $\epsilon^{BP}(\lambda,\rho)=\inf_{x\in(0,1]}\epsilon(x)$.

<u>HW</u> (Programming Exercise): Consider an LDPC code with n variable nodes and $\lambda(x)=x^2$, $\rho(x)=x^3$.

(i) Compute the degrees of the variable nodes and check nodes. What can you say about the ensemble LDPC(n,λ,ρ)?

(ii) Simulate the density evolution recursion for varying ϵ values and verify that for $\epsilon < \epsilon^{BP}(\lambda, \rho)$, we have $p_\ell \xrightarrow[\ell \to \infty]{} 0$.