# Codes for Input-Constrained Channels

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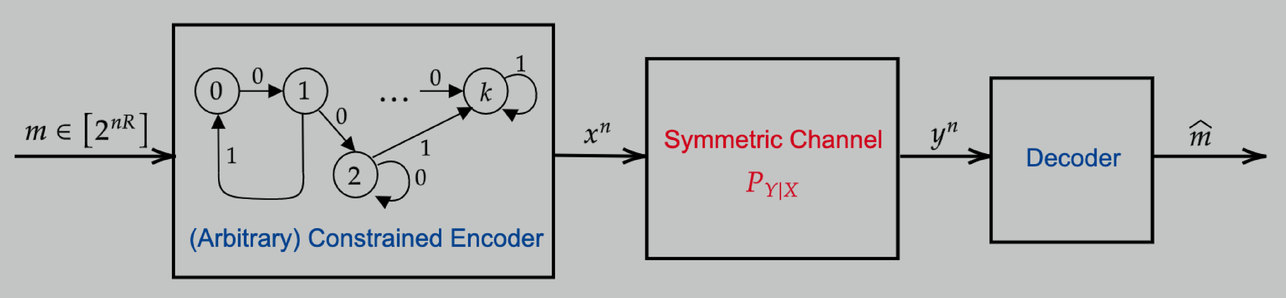
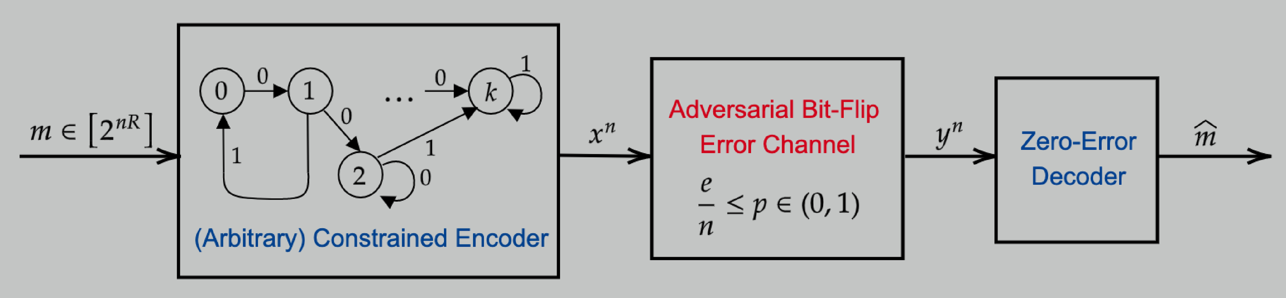
## Channel Models and Constraints

- Consider the setting of the transmission of information across input-constrained binary-input channels.
- We work with both stochastic and adversarial noise models.

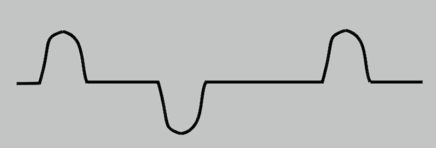


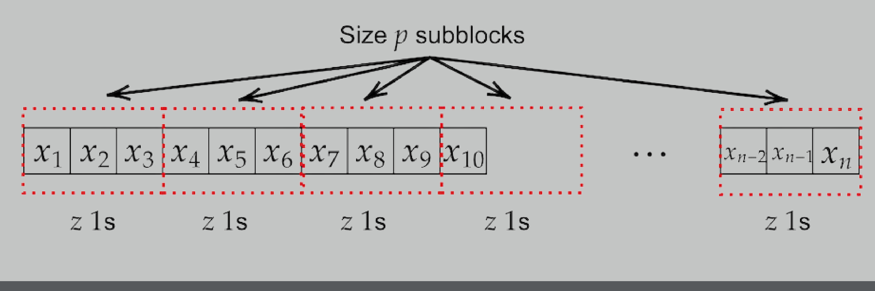

- The constraints we work with find application in a number of domains:

1. Runlength-limited (RLL) constraints: Alleviate ISI in magneto-optical recording

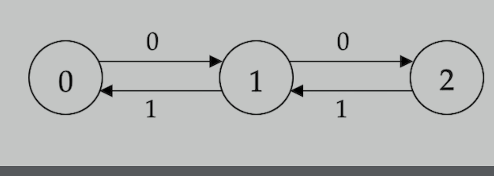
$\ldots 0100010000100 \ldots \longleftrightarrow$

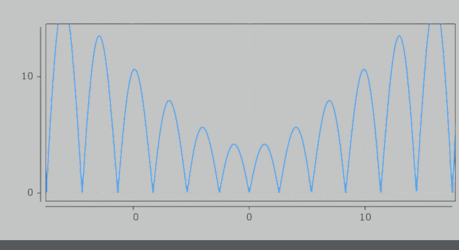


2. Subblock composition constraints: Aid in energy-harvesting in communication




3. Charge constraints: Ensure spectral nulls (DC-freeness) in frequency spectrum





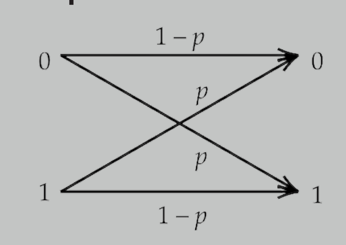
## Explicit Codes over RLL Input-Constrained Binary Memoryless Symmetric (BMS) Channels

- Our interest is in the $(d,\infty)$-RLL constraint, i.e., there must be at least $d$ **0**s between successive **1**s.

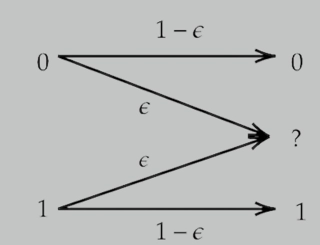
  Example: When $d = 2$,

  1 0 0 0 1 0 0 0 0 1 0 0 1 ✓
  1 0 0 1 0 1 0 0 0 1 0 0 1 ✗

- Examples of BMS channels:

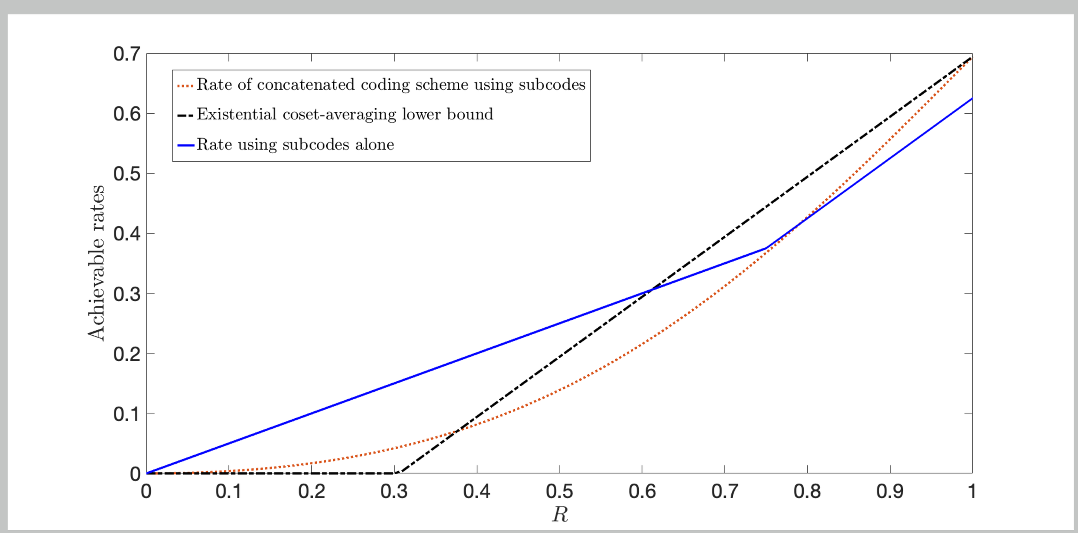


BSC with crossover probability $p$

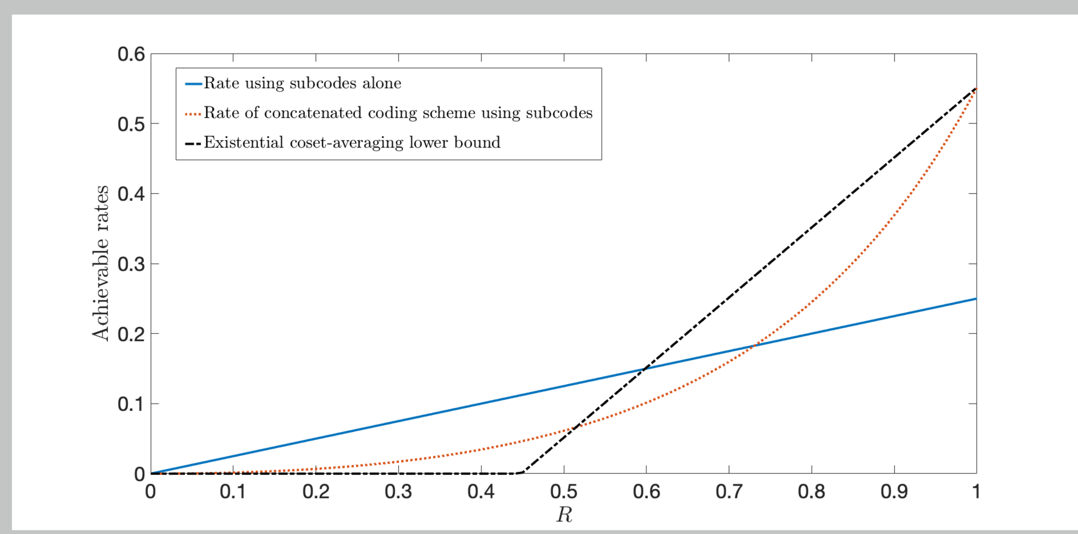


BEC with erasure probability $\epsilon$

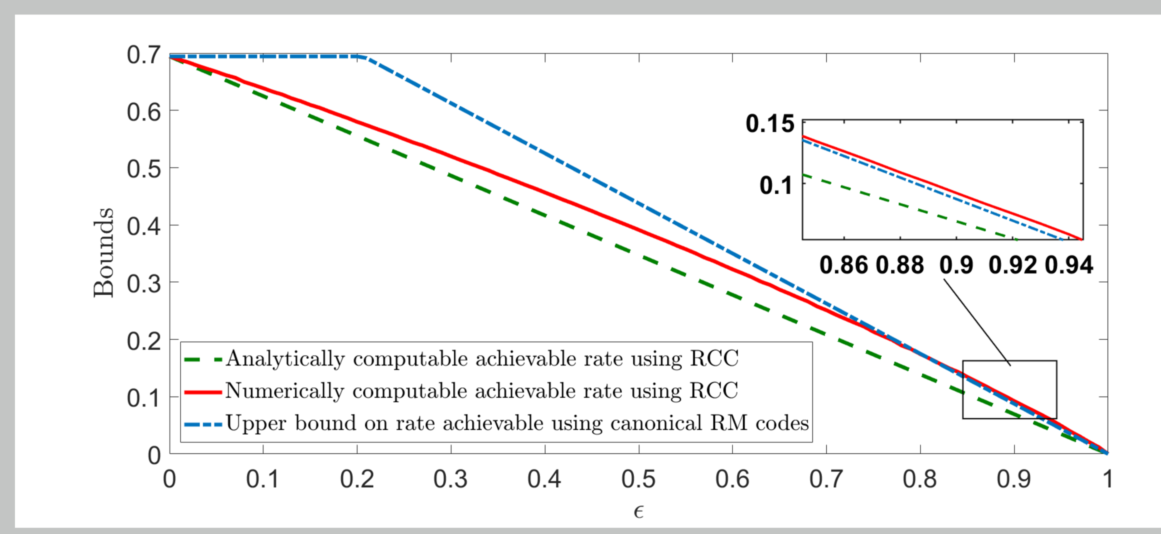
- We construct coding schemes, using Reed-Muller (RM) codes of rate $R$, over such input-constrained BMS channels of unconstrained capacity $C \in (0,1)$.




Rates achievable, when $d = 1$, if $R < C$




Rates achievable, when $d = 2$, if $R < C$




Rate bounds over the BEC($\epsilon$), when $d = 1$, using a canonical sequence of RM codes [RCC← Random Constrained Codes]

## Computing the Sizes of Constrained Subcodes of General Linear Codes

- Using a trick from Boolean Fourier analysis, we convert the problem into a question on the structure of the dual code:

$$N(\mathcal{C};\mathcal{A}) = \sum_{\mathbf{x}\in\{0,1\}^n} \mathbb{1}\{\mathbf{x}\in\mathcal{A}\}\cdot\mathbb{1}\{\mathbf{x}\in\mathcal{C}\} \overset{(Plancherel's)}{=} 2^n \cdot \sum_{\mathbf{s}\in\{0,1\}^n} \widehat{\mathbb{1}_{\mathcal{A}}}(\mathbf{s})\cdot\widehat{\mathbb{1}_{\mathcal{C}}}(\mathbf{s}) = |\mathcal{C}|\cdot\sum_{\mathbf{s}\in\mathcal{C}^\perp}\widehat{\mathbb{1}_{\mathcal{A}}}(\mathbf{s}).$$

$[\{0,1\}^n \supseteq \mathcal{A} \leftarrow$ Set of constrained sequences$]$

- For many constraints, the Fourier transform $\widehat{\mathbb{1}_{\mathcal{A}}}$ is computable!

1. $(d,\infty)$-RLL constraint ($S^d$):
   **Theorem:** For $n \ge d+2$ and for $\mathbf{s} = (s_1,\ldots,s_n) \in \{0,1\}^n$, it holds that
   $$\widehat{\mathbb{1}_{S^d}}^{(n)}(\mathbf{s}) = c_1\cdot\widehat{\mathbb{1}_{S^d}}^{(n-1)}(s_2^n) + c_2(s_1)\cdot\widehat{\mathbb{1}_{S^d}}^{(n-d-1)}(s_{d+2}^n),$$
   for $c_1, c_2(s_1) \in \mathbb{R}$.

2. Subblock composition constraint ($C_z^p$):
   **Theorem:** For $\mathbf{s} \in \{0,1\}^n$ with $\mathbf{s} = (\mathbf{s}_1 \mid \mathbf{s}_2 \mid \ldots \mid \mathbf{s}_p)$, we have that
   $$2^n \cdot \widehat{\mathbb{1}_{C_z^p}}(\mathbf{s}) = \prod_{\ell=1}^{p} \underbrace{K_z^{(n/p)}}_{\text{Krawtchouk poly.}}(w(\mathbf{s}_\ell)).$$

3. 2-charge constraint ($S_2$):
   **Theorem:** There exists a vector space $V$ such that for $\mathbf{s} \in V$,
   $$\widehat{\mathbb{1}_{S_2}}(\mathbf{s}) = (-1)^{\delta(\mathbf{s})}\cdot 2^{\lfloor \frac{n}{2}\rfloor - n},$$
   and $\widehat{\mathbb{1}_{S_2}}(\mathbf{s}) = 0$, otherwise, with $\delta(\mathbf{s}) \in \{0,1\}$.

## Upper Bounds on the Sizes of Constrained Codes Over Adversarial Channels

- The bit-flip error-correcting capability (with zero-error) of (constrained) codes is determined by the minimum Hamming distance, $d$.
- We propose a version of Delsarte' linear program for constrained systems, for upper bounding the sizes of constrained codes with a prescribed $d$.
- Our bounds beat the state-of-the-art generalized sphere packing bounds of Fazeli, Vardy, and Yaakobi (2015).

$$\underset{f:\,\{0,1\}^n\to\mathbb{R}}{\text{maximize}} \quad \sum_{\mathbf{x}\in\{0,1\}^n} f(\mathbf{x}) \qquad \text{(Obj}'\text{)}$$

subject to:

$f(\mathbf{x}) \ge 0,\ \forall\, \mathbf{x}\in\{0,1\}^n,$ (D1)
$\widehat{f}(\mathbf{s}) \ge 0,\ \forall\, \mathbf{s}\in\{0,1\}^n,$ (D2)
$f(\mathbf{x}) = 0,$ if $1 \le w(\mathbf{x}) \le d-1,$ (D3)
$f(0^n) \le \text{val}(\mathbf{Del}(n,d)),$ (D4)
$f(\mathbf{x}) \le 2^n\cdot(\mathbb{1}_{\mathcal{A}} \star \mathbb{1}_{\mathcal{A}})(\mathbf{x}),\ \forall\, \mathbf{x}\in\{0,1\}^n.$ (D5)

- Our LP $\mathbf{Del}(n,d;\mathcal{A})$ is given on the left, for any constraint represented by $\mathcal{A} \subseteq \{0,1\}^n$.
  - Delsarte's (unconstrained) LP is $\mathbf{Del}(n,d)$.
- Our upper bound is precisely $\text{val}(\mathbf{Del}(n,d;\mathcal{A}))^{1/2}$.
- $\mathbf{Del}(n,d;\mathcal{A})$ can be "symmetrized" to yield an LP with much smaller numbers of variables+constraints (sometimes only polynomially many!)
- It holds that
  $\text{val}(\mathbf{Del}(n,d;\mathcal{A}))^{1/2} \le \min\{|\mathcal{A}|, \text{val}(\mathbf{Del}(n,d))\}$.

| $d$ | Del($n=13, d; S_2$) | GenSph($n=13, d; S_2$) | Del($n=13, d$) |
|---|---|---|---|
| 3 | 45.255 | 64 | 512 |
| 4 | 45.255 | 64 | 292.571 |
| 5 | 22.627 | 64 | 64 |
| 6 | 17.889 | 64 | 40 |
| 7 | 5.657 | 32 | 8 |
| 8 | 4.619 | 32 | 5.333 |
| 9 | 2.828 | 16 | 3.333 |
| 10 | 2.619 | 16 | 2.857 |

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