Coding Schemes for Input-Constrained Channels

V. Arvind Rameshwar

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Indian Institute of Science, Bangalore

Networks Seminar 2023

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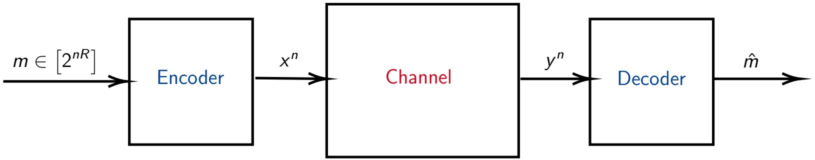
The big picture

We are broadly interested in the design of coding schemes that allow for reliable communication over noisy channels with memory.



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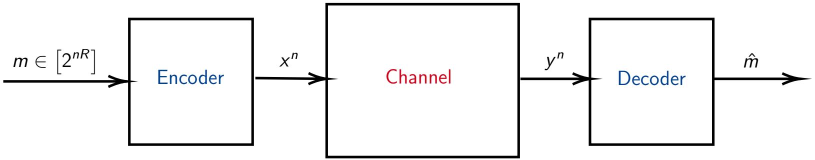

- The "memory" of the channel is encapsulated in a channel state s_i, at every time instant i, with the transitions between states (possibly) driven by the inputs.
- Examples: ISI channels, Gilbert-Elliott channels, input-constrained channels
- Our focus: Input-constrained channels

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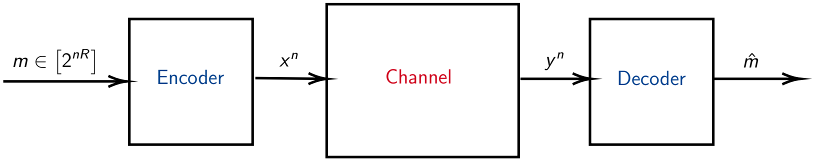

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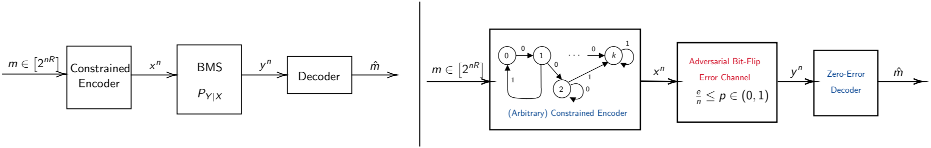
- Our focus: Input-constrained channels

Broad question: Can we design reliable coding schemes, with large rate $R \in (0, 1)$, over such channels?

Channel models and constraints: an overview

We consider the setting of the transmission of binary constrained codes over noisy channels.

- We consider both stochastic and combinatorial (worst-case) noise models.



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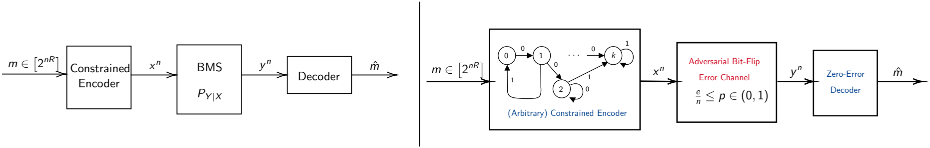

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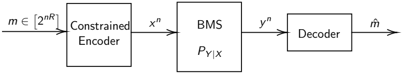
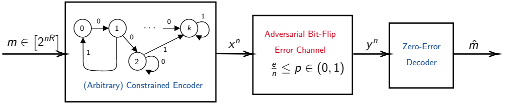


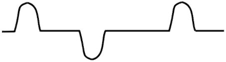


- The constrained sequences that we consider find application in a number of domains:

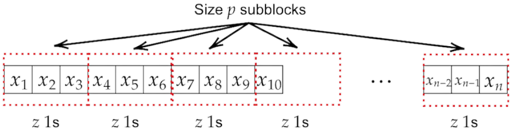
Some constraints of interest

- Runlength-limited (RLL) constraints: Help alleviate ISI in magneto-optical recording

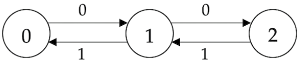
. . . 0 1 0 0 0 1 0 0 0 0 0 1 0 0 . . . $\longleftrightarrow$

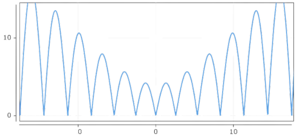


- Subblock composition constraints: Maintain receiver battery levels in energy-harvesting communication



- Charge constraints: Ensure spectral nulls (DC-freeness) in frequency spectrum





Talk outline

Part 1 Coding schemes over input-constrained symmetric channels via linear codes

A Explicit coding schemes over rulength-limited channels can be designed using Reed-Muller (RM) codes!

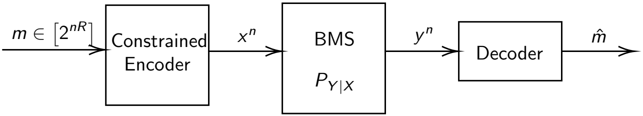
B A simple Fourier-analytic identity can help compute rates of arbitrarily constrained subcodes!

Part 2 Bounds on the resilience of constrained codes to worst-case (combinatorial) symmetric errors

- Delsarte's linear program (LP) can be extended to yield good bounds!

Part 1: Coding schemes over input-constrained symmetric channels

The channel model



We first focus on (stochastic) input-constrained Binary-Input Memoryless Symmetric (BMS) channels:

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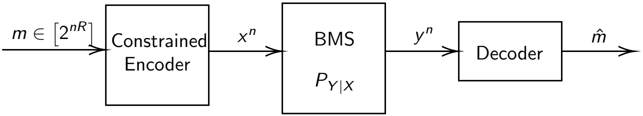

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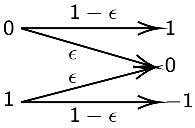
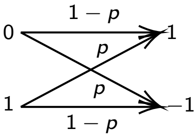


Binary Erasure Channel (BEC)



Binary Symmetric Channel (BSC)

Main idea and objectives

- Key Idea: Use explicit codes that achieve capacity over unconstrained BMS channels and select subcodes that comply with the input constraint.

- We know from [Reeves and Pfister (2022), Arikan (2009), Richardson and Urbanke (2001)] that there exist *linear codes* that achieve capacity over any BMS channel, under suitable decoding procedures.
  - Hence, *constrained subcodes* of such linear codes also enjoy vanishing error probabilities, under bit-MAP decoding.

- Goals:
  - Design explicit constrained coding schemes, using capacity-achieving linear codes, for select constraints.
  - Obtain estimates of the sizes of the largest constrained subcodes of general linear codes.

Part 1A: Constrained coding schemes using RM codes

The input constraint

The input constraint of interest to us is the (d, ∞)-runlength limited (RLL) input-constraint:

Definition

A binary sequence is said to satisfy the (d, ∞)-RLL constraint if there exist at least d 0s between every pair of successive 1s.

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- $(1, \infty)$-RLL $\equiv$ no-consecutive-ones.
- The (d, ∞)-RLL constraint on data sequences ensures that successive voltage peaks (or 1 bits) are spaced far apart, in order to alleviate ISI in magnetic recording systems.

Select prior art

Coding-theoretic work:

- Patapoutian and Kumar (1992): Coset-averaging lower bound on rates of constrained subcodes of cosets of linear codes.

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- Arnold et al. (2006), Han (2013, 2015), Li and Han (2018): Numerically computable achievable rates over general finite-state channels (FSCs), using Markov inputs
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In this part: Constrained code constructions using RM codes and explicit bounds on rates

Brief Background on RM Codes

- Codewords of RM codes consist of evaluation vectors of multivariate polynomials over $\mathbb{F}_2$.
- For a polynomial $f \in \mathbb{F}_2[x_1, x_2, \ldots, x_m]$ and a binary vector $\boldsymbol{z} = (z_1, \ldots, z_m)$, let $\text{Eval}_{\boldsymbol{z}}(f) := f(z_1, \ldots, z_m)$.
- Let the evaluation points $\boldsymbol{z}$ be ordered according to the standard lexicographic ordering:

$$000\ldots00 \to 000\ldots01 \to 000\ldots10 \to \ldots \to 111\ldots11$$

- We denote by $\text{Eval}(f) := (\text{Eval}_{\boldsymbol{z}}(f))_{\boldsymbol{z} \in \mathbb{F}_2^m}$, the 2^m-length vector of evaluations of f at points ordered in the lexicographic order

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Definition

The r^{th} order binary RM code $\text{RM}(m, r)$ is defined as the set of binary vectors:

$$\text{RM}(m, r) := \{\text{Eval}(f) : f \in \mathbb{F}_2[x_1, x_2, \ldots, x_m],\ \deg(f) \leq r\},$$

where $\deg(f)$ is the degree of the largest monomial in f and the degree of a monomial $x_S := \prod_{j \in S : S \subseteq [m]} x_j$ is simply $|S|$.

Brief Background on RM Codes

- Dimension and rate:

$$\dim(\mathrm{RM}(m,r)) = \#\{x_S \in \mathbb{F}_2[x_1,\ldots,x_m] : \ \deg(x_S) = |S| \le r\}$$
$$= \sum_{i=0}^{r} \binom{m}{i} =: \binom{m}{\le r}.$$

Hence, $\mathrm{rate}(\mathrm{RM}(m,r)) = \frac{\binom{m}{\le r}}{2^m}$.

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Hence, $\mathrm{rate}(\mathrm{RM}(m,r)) = \frac{\binom{m}{\le r}}{2^m}$.

- Example: For $R \in (0,1)$, consider the sequence of codes $\{\mathcal{C}_m(R) = \mathrm{RM}(m, r_m)\}_{m\ge 1}$, with

$$r_m := \max\left\{ \left\lfloor \frac{m}{2} + \frac{\sqrt{m}}{2} Q^{-1}(1-R) \right\rfloor, 0 \right\},$$

where

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\tau^2/2} d\tau, \ t \in \mathbb{R}.$$

- It can be checked that $\mathrm{rate}(\mathcal{C}_m) \xrightarrow{m\to\infty} R$.

Selected results

A simple construction using linear subcodes:

Theorem

For any $R \in [0, C)$, there exists a sequence of (d, ∞)-RLL linear subcodes $\{\mathcal{C}_m^{(d,\infty)}(R)\}$, where, for $t = \lceil \log_2(d+1) \rceil$,

$$\mathcal{C}_m^{(d,\infty)}(R) = \left\{ Eval(f) : f = \left(\prod_{i=m-t+1}^{m} x_i \right) \cdot h(x_1, \ldots, x_{m-t}), \; deg(h) \leq r_m - t \right\}.$$

Moreover, $rate\left(\mathcal{C}_m^{(d,\infty)}(R)\right) \xrightarrow{m \to \infty} R \cdot 2^{-\lceil \log_2(d+1) \rceil}$.

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Moreover, $rate\left(\mathcal{C}_m^{(d,\infty)}(R)\right) \xrightarrow{m \to \infty} R \cdot 2^{-\lceil \log_2(d+1) \rceil}$.

For example, when $d = 1$, these subcodes have all the symbols in odd coordinate positions as 0.

Selected results

Converse results for linear subcodes:

Theorem

For any sequence $\left\{\hat{\mathcal{C}}_m(R)\right\}_{m\geq 1}$ *of RM codes, following the lexicographic coordinate ordering, such that* $\text{rate}\left(\hat{\mathcal{C}}_m(R)\right) \xrightarrow{m\to\infty} R \in (0,1)$*, the largest rate of linear* (d,∞)*-RLL constrained subcodes is* $\frac{R}{d+1}$*.*

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We had previously identified (d,∞)-RLL linear subcodes $\left\{\mathcal{C}_m^{(d,\infty)}(R)\right\}_{m\geq 1}$ of rate $R \cdot 2^{-\lceil \log_2(d+1) \rceil} \approx \frac{R}{d+1}$.

Hence, the subcodes $\left\{\mathcal{C}_m^{(d,\infty)}(R)\right\}_{m\geq 1}$ are essentially rate-optimal.

Selected results

Converse results for linear subcodes (contd.):

Theorem

Under almost all coordinate orderings, the largest rate of linear (d, ∞)-RLL subcodes of RM codes of rate R, is at most $\frac{R}{d+1} + \epsilon$, for $\epsilon > 0$ arbitrarily small.

Existence results for non-linear subcodes:

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These subcodes are necessarily non-linear for $R > 0.75$, since then $R - \frac{3}{8} > \frac{R}{2}$.

Plots and Comparisons - I

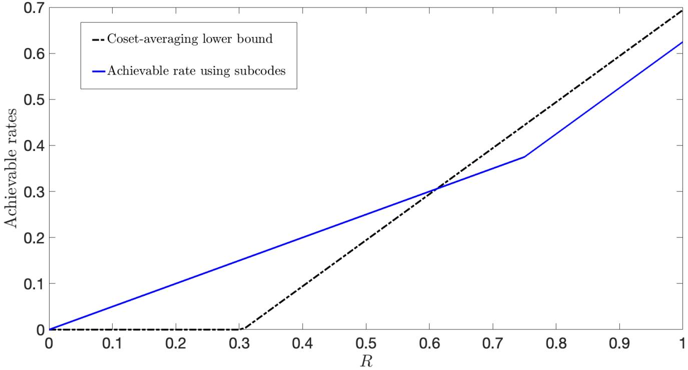


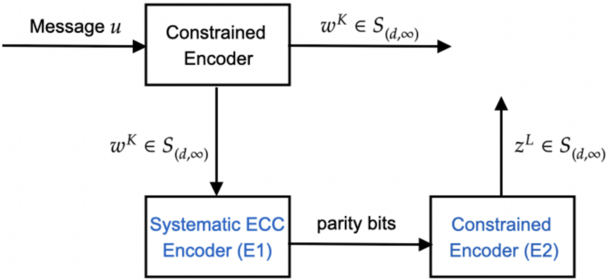
Figure: Plot comparing the achievable rates using $(1, \infty)$-RLL RM subcodes with the coset-averaging lower bound that is approximately $R - 0.3058$, of [Patapoutian and Kumar (1992)]

A concatenated coding scheme

We adopt the "reverse concatenation" strategy of [Bliss (1981)] and [Mansuripur (1991)] that is commonly used to limit error propagation during decoding of constrained codes.

A concatenated coding scheme

Encoding (the Bliss scheme):



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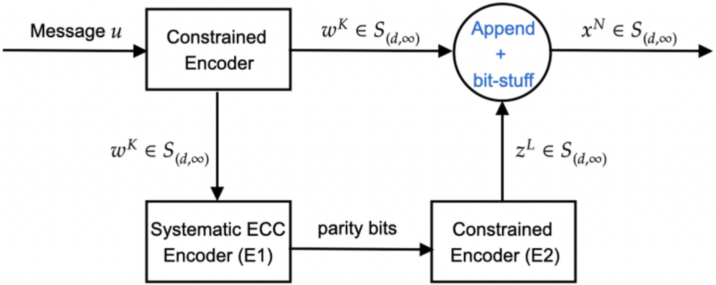

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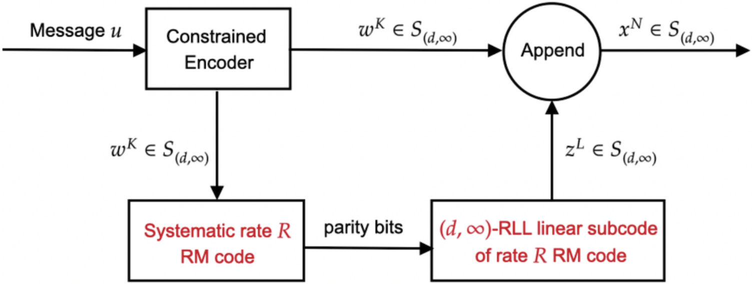

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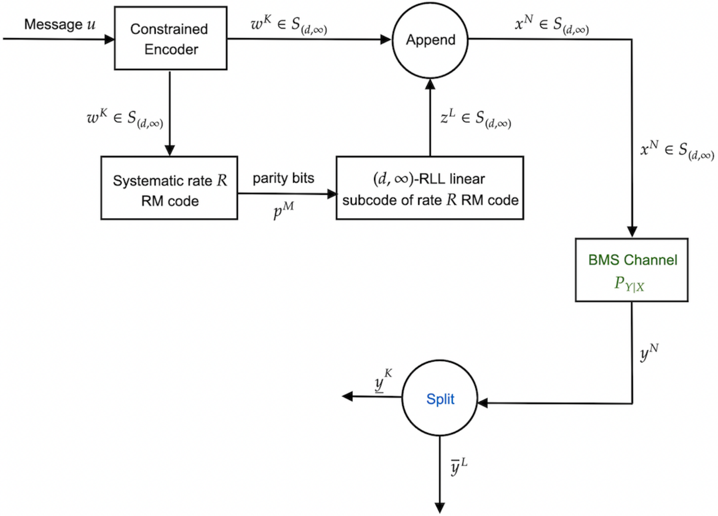
Encoding+Decoding:



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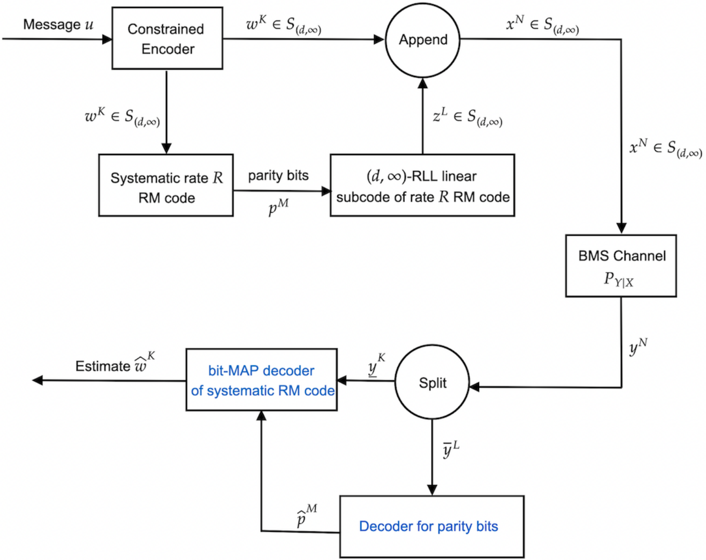

Coding theorem

Let C be the capacity of the unconstrained BMS channel.

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Theorem

For any $R \in (0, C)$, there exists a sequence of (d, ∞)-RLL constrained concatenated codes $\{\mathcal{C}_m^{conc}\}_{m \geq 1}$ that achieves a rate lower bound given by

$$\liminf_{m \to \infty} rate(\mathcal{C}_m^{conc}) \geq \frac{C_0^{(d)} \cdot R^2 \cdot 2^{-\lceil \log_2(d+1) \rceil}}{R^2 \cdot 2^{-\lceil \log_2(d+1) \rceil} + 1 - R + \epsilon},$$

over (d, ∞)-RLL input-constrained BMS channels, where $\epsilon > 0$ can be arbitrarily small.

Plots and Comparisons - II

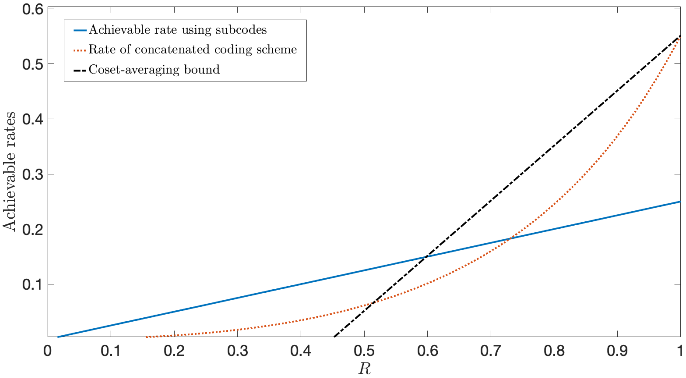


Figure: Plot comparing the achievable rates using $(2, \infty)$-RLL linear RM subcodes with the coset-averaging lower bound and the rate achieved by the concatenated coding scheme

Some questions

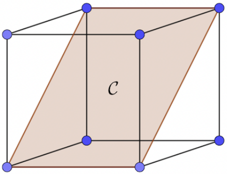
- Can we obtain better estimates of the weight distributions of RM codes, which will help sharpen our bounds?

- Can we extend the techniques in our work to design good codes over other finite-state channels such as Gilbert-Elliott channels (GECs) or ISI channels?

Part 1B: Counting constrained codewords in linear codes

The problem

- Motivated by the previous section, we now consider the problem of computing rates of (arbitrary) constrained codewords in general linear codes $\mathcal{C}$.



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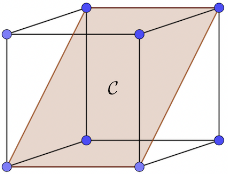

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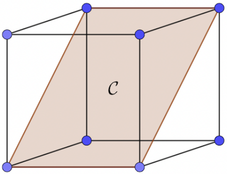

- Such an approach can help identify linear codes with large constrained subcodes.
- The problem: Given a set of constrained codewords $\mathcal{A} \subseteq \mathbb{F}_2^n$, we would like to gain insight into

$$N(\mathcal{C};\mathcal{A}) = \sum_{\boldsymbol{x}\in\mathcal{C}} \mathbb{1}\{\boldsymbol{x} \in \mathcal{A}\} = \sum_{\boldsymbol{x}\in\{0,1\}^n} \mathbb{1}\{\boldsymbol{x} \in \mathcal{A}\} \cdot \mathbb{1}\{\boldsymbol{x} \in \mathcal{C}\}.$$

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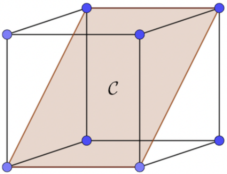

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This looks like an inner product between Boolean functions. . .

A brief refresher on Fourier analysis on $\mathbb{F}_2^n$

- Given any function $f : \{0,1\}^n \to \mathbb{R}$ and a vector $\boldsymbol{s} = (s_1, \ldots, s_n) \in \{0,1\}^n$, the Fourier coefficient of f at $\boldsymbol{s}$ is

$$\widehat{f}(\boldsymbol{s}) := \frac{1}{2^n} \sum_{\boldsymbol{x} \in \{0,1\}^n} f(\boldsymbol{x}) \cdot (-1)^{\boldsymbol{x} \cdot \boldsymbol{s}}.$$

- The functions $(\chi_{\boldsymbol{s}} : \boldsymbol{s} \in \{0,1\}^n)$, where $\chi_{\boldsymbol{s}}(\boldsymbol{x}) := (-1)^{\boldsymbol{x} \cdot \boldsymbol{s}}$, form a basis for the vector space V of functions $f : \{0,1\}^n \to \mathbb{R}$. Further, they are orthonormal with respect to the inner product $\langle \cdot, \cdot \rangle$, where:

$$\langle f, g \rangle := \frac{1}{2^n} \sum_{\boldsymbol{x} \in \{0,1\}^n} f(\boldsymbol{x}) g(\boldsymbol{x}).$$

Theorem (Plancherel's Theorem)

For any $f, g \in \{0,1\}^n \to \mathbb{R}$, we have that

$$\langle f, g \rangle = \sum_{\boldsymbol{s} \in \{0,1\}^n} \widehat{f}(\boldsymbol{s}) \widehat{g}(\boldsymbol{s}).$$

Workhorse

Observe that

$$N(\mathcal{C};\mathcal{A}) = \sum_{\boldsymbol{x}\in\{0,1\}^n} \mathbb{1}\{\boldsymbol{x}\in\mathcal{A}\}\cdot\mathbb{1}\{\boldsymbol{x}\in\mathcal{C}\}$$

$$= 2^n \cdot \sum_{\boldsymbol{s}\in\{0,1\}^n} \widehat{\mathbb{1}_{\mathcal{A}}}(\boldsymbol{s})\cdot\widehat{\mathbb{1}_{\mathcal{C}}}(\boldsymbol{s}).$$

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For linear codes $\mathcal{C}$, it is easy to show that

$$\widehat{\mathbb{1}_{\mathcal{C}}}(\boldsymbol{s}) = \frac{|\mathcal{C}|}{2^n}\cdot\mathbb{1}_{\mathcal{C}^\perp}(\boldsymbol{s}).$$

Hence, we have that

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Observations:

1. If $\dim(\mathcal{C}) \gg n/2$, then we can employ our insight to count over a low-dimensional space!
2. For many constraints of interest, the Fourier transform above is computable!

Example 0: warm-up

- Consider the constant weight constraint that admits only binary sequences of a fixed weight $i \in [0 : n]$ (we let W_i denote the set of such words). Let us write $a_i(\mathcal{C}) := N(\mathcal{C}; W_i)$.
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$$a_i(\mathcal{C}) = \frac{1}{|\mathcal{C}^\perp|} \sum_{j=0}^{n} K_i^{(n)}(j) \cdot a_j(\mathcal{C}^\perp),$$

where $K_i^{(n)}(x)$ is the i^{th}-Krawtchouk polynomial at length n, with $K_i^{(n)}(z) = \sum_{\ell=0}^{i} (-1)^\ell \binom{z}{\ell} \binom{n-z}{i-\ell}$.

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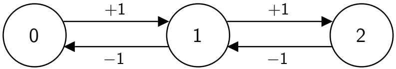
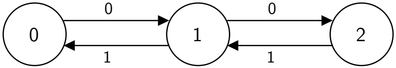
These are simply MacWilliams' identities for linear codes.

Example 1: 2-charge constraint

- We now consider a spectral null constraint, whose sequences in $\{+1,-1\}^n$ have a null at zero frequency (sometimes called a DC-free constraint). Our constraint is the so-called 2-charge constraint.



- We let S_2 denote those sequences in $\{0,1\}^n$ that can be mapped to 2-charge constrained sequences via the map $x \mapsto (-1)^x$, for $x \in \{0,1\}$.



Sequences in S_2 can be read off the labels of paths in the above graph.

Computation of Fourier coefficients

- We define the set of vectors (when n is odd)

$$\boldsymbol{b}_0 := 100\ldots00, \qquad \boldsymbol{b}_1 := 0\underbrace{11}00\ldots00,$$

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Theorem

For $\boldsymbol{a} = \left(a_0, a_1, \ldots, a_{\lceil\frac{n}{2}\rceil-1}\right) \in \{0,1\}^{\lceil\frac{n}{2}\rceil}$, *consider* $\boldsymbol{s} = \sum_{i=0}^{\lceil\frac{n}{2}\rceil-1} a_i \cdot \boldsymbol{b}_i$. *It holds that*

$$\widehat{\mathbb{1}_{S_2}}(\boldsymbol{s}) = 2^{\lfloor\frac{n}{2}\rfloor-n} \cdot (-1)^{w(\boldsymbol{a})-a_0}.$$

Further, for $\boldsymbol{s} \notin V_{\mathcal{B}}$, *we have that* $\widehat{\mathbb{1}_{S_2}}(\boldsymbol{s}) = 0$.

Some consequences

Consider now the following condition (**C**):

$$(\mathbf{C}) \quad \text{For all } \boldsymbol{s} \in \mathcal{C}^{\perp} \cap V_{\mathcal{B}}, \text{ it holds that } \widehat{\mathbb{1}_{S_2}}(\boldsymbol{s}) \geq 0.$$

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Then, applying our approach and using the previous theorem, we get that

- If condition (**C**) is not satisfied, then, $N(\mathcal{C}; S_2) = 0$.
- If condition (**C**) is satisfied and there exist $t_n \in \left[1 : \left\lceil \frac{n}{2} \right\rceil - 1\right]$ linearly independent vectors $(\boldsymbol{s}_1, \ldots, \boldsymbol{s}_{t_n})$ in $\mathcal{C}^{\perp}$ with $\widehat{\mathbb{1}_{S_2}}(\boldsymbol{s}_i) > 0$, for all $1 \leq i \leq t_n$, then, $N(\mathcal{C}; S_2) = |\mathcal{C}| \cdot 2^{t_n + \left\lfloor \frac{n}{2} \right\rfloor - n}$.

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Hence, it is possible to construct a sequence $\{\mathcal{C}^{(n)}\}_{n \geq 1}$ of linear codes of rate R such that the rate of their constrained subcodes is

$$\liminf_{n \to \infty} \text{rate}\left(\mathcal{C}_2^{(n)}\right) = R - \frac{1}{2} + \liminf_{n \to \infty} \frac{t_n}{n}.$$

Therefore, we can obtain rates better than the coset-averaging bound of [Patapoutian and Kumar (1992)], using subcodes!

Applications to specific linear codes

We also applied our approach to count the number of words in S_2, in specific linear codes.

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Hamming codes:

Theorem

For $m \geq 3$ and for $\mathcal{C}$ being the $[2^m - 1, 2^m - 1 - m]$ Hamming code under a canonical coordinate ordering, we have that

$$N(\mathcal{C}; S_2) = 2^{\left\lfloor \frac{2^m - 1}{2} \right\rfloor - 1}.$$

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Reed-Muller codes:

(m, r)	$(5, 3)$	$(6, 4)$	$(7, 5)$	$(8, 6)$
$N(\mathrm{RM}(m, r); S_2)$	2048	6.711×10^7	1.441×10^{17}	1.329×10^{36}

Some sample numerical values for high rate RM codes

Example 2: (d, ∞)-RLL constraint

- We return to our familiar runlength-limited constraint, which requires that there be at least d 0s between successive 1s. Let S^d denote the set of constrained sequences.

- Let $\widehat{\mathbb{1}_{S^d}}^{(n)}$ denote the Fourier transform at blocklength $n \geq 1$.

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Theorem

For $n \geq d+2$ and for $\mathbf{s} = (s_1, \ldots, s_n) \in \{0,1\}^n$, it holds that

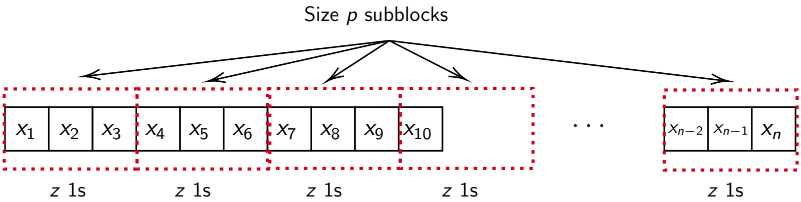
$$\widehat{\mathbb{1}_{S^d}}^{(n)}(\mathrm{s}) = 2^{-1} \cdot \widehat{\mathbb{1}_{S^d}}^{(n-1)}(s_2^n) + (-1)^{s_1} \cdot 2^{-(d+1)} \cdot \widehat{\mathbb{1}_{S^d}}^{(n-d-1)}\left(s_{d+2}^n\right).$$

- The above theorem gives rise to a recursive algorithm for computing $\widehat{\mathbb{1}_{S^d}}^{(n)}$.

This recursive procedure is faster than the Fast Walsh-Hadamard Transform!

Example 3: Subblock-composition constraints

- Next, consider the so-called subblock composition constraint, which admits binary sequences that have a fixed number of 1s in each "subblock".
  - Application: In energy-harvesting communication, ensures that the receiver's battery does not drain out during periods of low symbol energy[1].



- Let C_z^p denote the set of such constrained sequences.

[1] A. Tandon, M. Motani, and L. R. Varshney, "Subblock-constrained codes for real-time simultaneous energy and information transfer," T-IT, 2016.

Computation of Fourier coefficients

- Standard arguments lead us to the characterization of Fourier coefficients:

Theorem

For $\mathbf{s} \in \{0,1\}^n$ with $\mathbf{s} = \mathbf{s}_1\mathbf{s}_2\ldots\mathbf{s}_p$, we have that

$$2^n \cdot \widehat{\mathbb{1}_{C_z^p}}(\mathbf{s}) = \prod_{\ell=1}^{p} K_z^{(n/p)}(w(\mathbf{s}_\ell)),$$

where $K_i^{(n/p)}(j) = \sum_{t=0}^{i}(-1)^t\binom{j}{t}\binom{n/p-j}{i-t}$ is the i^{th}-Krawtchouk polynomial, for the length n/p.

[2] V. A. R. and N. Kashyap, "Estimating the sizes of binary error-correcting constrained codes," submitted to the IEEE JSAIT.

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- We show[2] that this characterization can help speed up the computation of constrained subcodes of RM codes, for select values of p.

[2] V. A. R. and N. Kashyap, "Estimating the sizes of binary error-correcting constrained codes," submitted to the IEEE JSAIT.

Some questions

- Can we come up with recursive algorithms for efficient computation of the Fourier coefficients for a large class of constraints (say, finite-type constraints, representable by a finite, labelled, directed graph)?

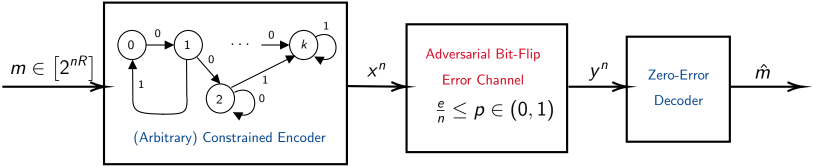
- Can we obtain estimates of the asymptotic rates of constrained subcodes of specific linear codes, using our methods?

- Can we find more use cases of a Fourier-analytic approach to counting constrained codewords?
  - For example, our Fourier coefficients can also be used to compute the weight distributions of constrained words in $\mathbb{F}_2^n$ and in a given linear code $\mathcal{C}$.

Part 2: A version of Delsarte's LP for constrained systems

The setting

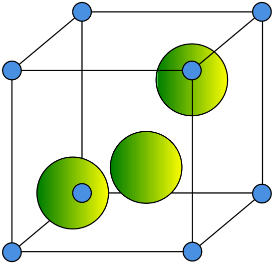
- Consider the channel model where the constrained codewords we intend transmitting, are subjected to adversarial (or worst-case) bit-flip errors (or erasures).



- We wish to uniquely recover the transmitted codeword, with zero error.
- Equivalently, we wish to come up with bounds on the sizes of constrained codes with a given minimum Hamming distance.

The problem

- Given a length-n constrained system represented by a set $\mathcal{A}$ of constrained words, what is the largest size of a constrained code with minimum distance at least d?



- This is a packing problem for which there exist generalized sphere packing bounds [Fazeli, Vardy, Yaakobi (2015), Cullina, Kiyavash (2016)]

Can we improve on these bounds?

Delsarte's LP (for unconstrained systems)

Let $A(n, d)$ denote the size of the largest (not necessarily linear) length-n code with minimum distance at least d.

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Consider the following LP L:

$$\begin{aligned}
\text{maximize} \quad & \sum_{w=0}^{n} a_w \\
\text{subj. to} \quad & a_w \geq 0, \text{ for all } w \in [0:n], \\
& \sum_{j=0}^{n} a_j \cdot K_w(j) \geq 0, \text{ for all } w \in [0:n], \\
& a_w = 0, \text{ for } w \in [1:d-1], \\
& a_0 = 1.
\end{aligned}$$

- Delsarte (1973) proved that the distance distribution $(b_w : 0 \leq w \leq n)$ of *any* binary length-n code $\mathcal{C}$ of minimum distance at least d is a feasible solution to the LP above, with objective value $|\mathcal{C}|$.
- Hence, $A(n,d) \leq \text{val}(\text{L})$.

Taking a step back: an alternative formulation

Del(n, d)

$$\underset{f\colon \{0,1\}^n \to \mathbb{R}}{\text{maximize}} \quad \sum_{\boldsymbol{x} \in \{0,1\}^n} f(\boldsymbol{x})$$

subject to:

$$f(\boldsymbol{x}) \geq 0, \ \forall \ \boldsymbol{x} \in \{0,1\}^n,$$

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- A feasible solution to the above LP is $\frac{2^n}{|\mathcal{C}|} \cdot (\mathbb{1}_{\mathcal{C}} \star \mathbb{1}_{\mathcal{C}})$, for $\mathcal{C}$ of minimum distance at least d. Here, for $f, g : \{0,1\}^n \to \mathbb{R}$,

$$(f \star g)(\boldsymbol{x}) := \frac{1}{2^n} \sum_{\boldsymbol{z} \in \{0,1\}^n} f(\boldsymbol{z}) \cdot g(\boldsymbol{x} + \boldsymbol{z}).$$

- The objective value again is $|\mathcal{C}|$.

Our LP for constrained systems

Let $A(n, d; \mathcal{A})$ denote the largest length-n $\mathcal{A}$-constrained code of minimum distance at least d.

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$$f(0^n) \leq \mathrm{val}(\mathrm{Del}(n, d)),$$

$$f(\boldsymbol{x}) \leq 2^n \cdot (\mathbb{1}_{\mathcal{A}} \star \mathbb{1}_{\mathcal{A}})(\boldsymbol{x}), \ \forall \ \boldsymbol{x} \in \{0,1\}^n.$$

- A feasible solution to $\mathrm{Del}(n, d; \mathcal{A})$ can again be constructed, whose objective value is $(A(n, d; \mathcal{A}))^2$.

How good is the LP?

- We ran our LP on the $(1,\infty)$-RLL constrained system ($n = 10$) ...

d	$\text{Del}(n, d; S_{(1,\infty)})$	$\text{GenSph}(n, d; S_{(1,\infty)})$	$\text{Del}(n, d)$
2	128.557	144	512
3	74.762	111	85.333
4	42.048	111	42.667
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- Our LP appears to perform better than the generalized sphere packing upper bounds.

Some comments

- The LP $\mathrm{Del}(n, d; \mathcal{A})$ as stated has 2^n variables for a blocklength n. For select constraints, it is indeed possible to "symmetrize" the LP using the symmetry group of the constraint (i.e., the group of index permutations that leave $\mathbb{1}_{\mathcal{A}}$ unchanged).

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- Future work can apply modern Fourier analytic techniques à la [Navon, Samorodnitsky (2005)], [Loyfer, Linial (2022)] or expander graph-related tools [Friedman, Tillich (2005)] to obtain asymptotic rate-distance tradeoff upper bounds using our LP.

Some questions

- Can we derive asymptotic upper bounds on the rate-distance tradeoff for constrained codes, using our LP formulation?

- Can we connect our Fourier-analytic techniques of counting, with this packing problem?

Thank You!